Pre-BC Topics Solutions

1. (a) Notice that $x^2 - x = x(x-1)$. Thus, we can clear the denominators by multiplying both sides by $x(x-1)$. Next, set the coefficients of the corresponding powers of x equal and solve the resulting system:

$$8x - 5 = Ax + B(x-1) \Rightarrow 8x - 5 = (A+B)x - B \Rightarrow \begin{cases} A + B = 8 \\ B = 5 \end{cases}$$

so the solution is $\boxed{A = 3, B = 5}$.

(b) Since $x^2 - 1 = (x-1)(x+1)$, we can clear the denominators by multiplying both sides by $(x-1)(x+1)$. This yields

$$x^2 + 2x - 1 = x^2 - 1 + Cx + C + Dx - D$$
$$2x = (C + D + 1)x + C - D$$
$$\begin{cases} C + D = 2 \\ C - D = 0 \end{cases} \Rightarrow 2C = 2 \Rightarrow \boxed{C = 1} \Rightarrow \boxed{D = 1}.$$

(c) Notice that $x^2 - x - 6 = (x-3)(x-2)$. Clear the denominators by multiplying both sides by $(x-3)(x+2)$ to get

$$2x^2 + x + 9 = 2x^2 - 2x - 12 + Ex + 2E + Fx - 3F$$

$$3x + 21 = (E+F)x + 2E - 3F$$

$$\begin{cases} E + F = 3 \\ 2E - 3F = 21 \end{cases} \Rightarrow \begin{cases} F = 3 - E \\ 2E - 3(3-E) = 21 \end{cases} \Rightarrow 5E - 9 = 21 \Rightarrow \boxed{E = 6} \Rightarrow \boxed{F = -3}$$

2. Let the difference between consecutive terms be d. Then, the fourth term is $7 + 3d$ and the fifth term is $7 + 4d$. So, $(7+3d) + (7+4d) = 42 \Rightarrow 14 + 7d = 42 \Rightarrow d = 4$. Therefore, the sixth term is $7 + 5d = \boxed{27}$.

3. We will use the sum of an arithmetic sequence formula, $S_k = \dfrac{k}{2}(a_1 + a_k)$, where S_k is the sum of the first k terms, a_1 is the first term and a_k is the k^{th} term. The total number of miles Ana walks on the k^{th} day is $\dfrac{k}{2}(0.5 + 0.5(k+1)) = \dfrac{k}{2}(0.5k + 1) = \dfrac{0.5k^2 + k}{2}$. Thus, we want to find the smallest positive integer that satisfies the inequality $\dfrac{0.5k^2 + k}{2} \geq 100 \Rightarrow k^2 + 2k \geq 400$. $k = \boxed{20}$ clearly works and $k = 19$ does not, since $19^2 + 2 \cdot 19 = 399 < 400$.

4. Let the first term of the sequence be a and the common ratio be r. Then ar is the second term and ar^3 is the fourth term, so $ar = 18$ and $ar^3 = \dfrac{81}{2}$. Solving for r yields $r^2 = \dfrac{9}{4} \Rightarrow r = \pm\dfrac{3}{2}$. This means that there are two possible answers to the problem. Solving $ar = 18$ for a, we see that a can be 12 or -12 depending on the sign of r. It follows that the sum of the first three terms is

$$a + ar + ar^2 = 12 + 18 + 27 = \boxed{57}$$

or

$$a + ar + ar^2 = -12 + 18 - 27 = \boxed{-21}.$$

5. (a) Let d be the common difference. Then, $a_2 = a_1 + d = 7$ and $a_7 = a_1 + 6d = 22$. We can find a_1 and d, find the first six terms and then find their sum. We first solve the system

by elimination. We can subtract the first equation from the second equation:

$$\begin{cases} a_1 + d = 7 \\ a_1 + 6d = 22 \end{cases} \Rightarrow 5d = 15 \Rightarrow d = 3 \Rightarrow a_1 = 7 - d = 4.$$

Thus, the first six terms are $4, 7, 10, 13, 16, 19$, which have a sum of $\boxed{69}$. Note that the sum of the first six terms can also be calculated by using the formula $\frac{n}{2}(2a_1 + (n-1)d)$, where n is the number of terms.

(b) The first four terms of this sequence are $5, 15, 45, 135$. These have a sum of $\boxed{200}$. Note that this can also be calculated using the formula $a_1\left(\frac{1-r^n}{1-r}\right)$, where n is the number of terms.

(c) Note that this is a geometric series with the first term 20 and the common ratio $\frac{2}{3}$. The sum of an infinite geometric series with first term a_1 and common ratio r is $\frac{a_1}{1-r}$, so the sum of the series in the problem is $\frac{20}{1-\frac{2}{3}} = \boxed{60}$.

6. The price of bread each day can be modeled using a geometric sequence with $a_1 = 5$ and common ratio $r = 1.1$. The sum of a geometric series with first term a_1, common ratio r, and number of terms n is $a_1\left(\frac{1-r^n}{1-r}\right)$. Thus, we need to find the least positive integer n satisfying the inequality

$$\begin{aligned} a_1\left(\frac{1-r^n}{1-r}\right) &\geq 100 \\ 5\left(\frac{1-(1.1)^n}{1-1.1}\right) &\geq 100 \\ 1 - 1.1^n &\leq -2 \\ 1.1^n &\geq 3 \\ n &\geq \log_{1.1} 3 \approx 11.5. \end{aligned}$$

It will take $\boxed{12}$ days for Alex to spend more than \$100.

7. (a) Recall the conversion formulas $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$. Note that $r = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} =$

$\sqrt{4} = 2$. Also $\tan\theta = 1$ and the point $(\sqrt{2}, \sqrt{2})$ is in the first quadrant, so θ is also in the first quadrant. Therefore, $\theta = \tan^{-1}(1) = \frac{\pi}{4}$. Therefore, $(\sqrt{2}, \sqrt{2}) = \boxed{\left(2, \frac{\pi}{4}\right)}$. Notice that there are many other ways to represent $(\sqrt{2}, \sqrt{2})$ using polar coordinates, such as $\left(-2, \frac{5\pi}{4}\right)$ and $\left(2, \frac{9\pi}{4}\right)$. A point can be represented uniquely in polar form if $r > 0$ and $0 \leq \theta < 2\pi$ and the origin is represented by $(0, 0)$.

(b) The conversion formulas yield $r = \sqrt{(-500)^2 + (500\sqrt{3})^2} = \sqrt{4 \cdot 500^2} = 1000$. Next, we note that $\tan\theta = -\sqrt{3}$, and since θ is in the second quadrant, $\theta = \pi + \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$. Therefore, $(-500, 500\sqrt{3}) = \boxed{\left(1000, \frac{2\pi}{3}\right)}$.

(c) The conversion formulas yield $r = \sqrt{\left(\frac{17\sqrt{3}}{2}\right)^2 + \left(-\frac{17}{2}\right)^2} = 17$. Next, we note that $\tan\theta = -\frac{\sqrt{3}}{3}$ and since θ is in the fourth quadrant, $\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6} = \frac{11\pi}{6}$.. Therefore, $\left(\frac{17\sqrt{3}}{2}, -\frac{17}{2}\right) = \boxed{\left(17, \frac{11\pi}{6}\right)}$.

8. (a) We use the conversion formulas $x = r\cos\theta, y = r\sin\theta$. Thus $x = 5\cos 120° = -\frac{5}{2}$ and $y = 5\sin 120° = \frac{5\sqrt{3}}{2}$, so $(5, 120°) = \boxed{\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)}$.

(b) Conversion formulas yield $x = 3\sqrt{2}\cos\frac{\pi}{4} = 3$ and $y = 3\sqrt{2}\sin\frac{\pi}{4} = 3$. It follows that $\left(3\sqrt{2}, \frac{\pi}{4}\right) = \boxed{(3, 3)}$.

(c) Conversion formulas yield $x = 6\cos\frac{4\pi}{3} = -3$ and $y = 6\sin\frac{4\pi}{3} = -3\sqrt{3}$, so $\left(6, \frac{4\pi}{3}\right) = \boxed{(-3, -3\sqrt{3})}$.

9. We use the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. Substitution yields

$$(r\cos\theta - 1)^2 + (r\sin\theta - 3)^2 = 4 \Rightarrow r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta + 1 - 6r\sin\theta + 9 = 4$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can further simplify for a final answer of $\boxed{r^2 - 2r\cos\theta - 6r\sin\theta + 6 = 0}$.

10. Note that $x^2 + y^2 = r^2$. Using this in conjunction with the fact that $x = r\cos\theta$ and $y =$

$r\sin\theta$, we obtain

$$x^2+y^2+4x+6y-3=0 \Rightarrow \boxed{(x+2)^2+(y+3)^2=16}$$

11. (a) Solving for y, we get that $y=-2x+4$. Thus, $4x+2y=8$ is equivalent to $\boxed{x=t, y=-2t+4}$. We could have also solved for x to get that $x=\dfrac{4-y}{2}$, so another parametrization is $x=\dfrac{4-t}{2}, y=t$. There are many other parametrizations.

    (b) The key to solving this problem is to realize that it is useful to use trigonometric functions when representing circles. Note that $\cos^2 t+\sin^2 t=1$, so $16\cos^2 t+16\sin^2 t=16$. We observe that we can let $(x-5)^2$ be equal to $16\cos^2 t$, and similarly $(y+3)^2$ be equal to $16\sin^2 t$. Solving for x, we get that $x-5=\pm 4\cos t \Rightarrow x=5\pm 4\cos t$. Solving for y, we get that $y+3=\pm 4\sin t \Rightarrow y=-3\pm 4\sin t$. A possible answer is $\boxed{x=5+4\cos t, y=-3+4\sin t}$. We could have chosen the signs differently. For example, $x=5+4\cos t, y=-3-4\sin t$ results in the same circle.

    (c) We can use trigonometric functions to parametrize an ellipse. Note that $\cos^2 t+\sin^2 t=1$, so $25\cos^2 t+25\sin^2 t=25$. We observe that we can let $\dfrac{x^2}{4}$ be equal to $25\cos^2 t$, and similarly $\dfrac{y^2}{9}$ be equal to $25\sin^2 t$. Solving for x, we get that $x^2=100\cos^2 t \Rightarrow x=\pm 10\cos t$. Solving for y, we get that $y^2=225\sin^2 t \Rightarrow y=\pm 15\sin t$. It follows that a possible answer is $\boxed{x=10\cos t, y=15\sin t}$. Other possible answers include, for example, $x=-10\cos t, y=15\sin t$ and $x=10\cos t, y=-15\sin t$.

12. The graph is a circle of radius 2 centered at $(0,6)$, which is equivalent to the equation $x^2+(y-6)^2=4$. Note that $\cos^2 t+\sin^2 t=1$, so $4\cos^2 t+4\sin^2 t=4$. We observe that we can let x^2 be equal to $4\cos^2 t$, and similarly $(y-6)^2$ be equal to $4\sin^2 t$. Taking square roots in each equation yields $x=\pm 2\cos t, y=6\pm\sin t$. One possible parametrization is $\boxed{x=2\cos t, y=6+2\sin t}$.

13. Note that the unit circle can be parametrized by $x=\cos t, y=\sin t$ and if t is measured in seconds, a particle with coordinates $(\cos t, \sin t)$ will travel around the circle counterclock-

wise and complete one full rotation after 2π seconds – the particle is at $(1,0)$ at $t=0$ and the next time it will be at $(1,0)$ when $t=2\pi$. A parametrization for a particle traversing a circle with radius 2 in 2π seconds would then be $(2\cos t, 2\sin t)$. If we wish to change the period to 10, the parametrization would be $\left(2\cos\left(\frac{2\pi}{10}t\right), 2\sin\left(\frac{2\pi}{10}t\right)\right)$. If we now wish to change the center of the circle from $(0,0)$ to $(3,4)$, the new parametrization would be $\boxed{x=3+2\cos\left(\frac{t\pi}{5}\right), y=4+2\sin\left(\frac{t\pi}{5}\right)}$. Note that at $t=0$ the particle is at $(5,4)$, which is precisely the condition in the problem, so we don't need to introduce a phase shift.

14. (a) $-\vec{x}+4\vec{y}=\langle -5,3\rangle+\langle 24,40\rangle=\langle -5+24, 3+40\rangle=\boxed{\langle 19,43\rangle}$.

(b) $2\vec{x}-(\vec{y}+\langle 2,7\rangle)=\langle 10,-6\rangle-\langle 6,10\rangle-\langle 2,7\rangle=\boxed{\langle 2,-23\rangle}$.

(c) $\langle 5,-3\rangle\cdot\langle 6,10\rangle=5\cdot 6+(-3)\cdot 10=\boxed{0}$. Note that this also shows that $\vec{x}$ and $\vec{y}$ are perpendicular.

(d) If the desired vector is parallel to $\langle 5,-3\rangle$ then it can be expressed as $\langle 5k,-3k\rangle$. Since its magnitude is 8, $\sqrt{(5k)^2+(-3k)^2}=8$, or $34k^2=64$, so $k=\pm\sqrt{\frac{64}{34}}=\pm\frac{4\sqrt{34}}{17}$. Therefore, the vector is $\boxed{\pm\left\langle \frac{20\sqrt{34}}{17}, -\frac{12\sqrt{34}}{17}\right\rangle}$.

15. To calculate the resultant force, we need to add the two force vectors. We can express each vector as a sum of its horizontal and vertical components, which will make the addition more straightforward. Note that if a vector has magnitude r and forms angle θ with the positive x-axis, measured counterclockwise, then $\vec{v}=\langle r\cos\theta, r\sin\theta\rangle$. This is similar to converting from rectangular to polar coordinates.

The force of 100N makes a 30° angle with the positive x-axis, so expressed as a vector, it is equal to $\langle 100\cos 30°, 100\sin 30°\rangle=\langle 50\sqrt{3},50\rangle$. The other force of 150N can be represented as $\langle 150\cos 150°, 150\sin 150°\rangle=\left\langle -75\sqrt{3},75\right\rangle$. Since the vectors sum to

$$\langle 50\sqrt{3},50\rangle+\left\langle -75\sqrt{3},75\right\rangle=\left\langle -25\sqrt{3},125\right\rangle,$$

the magnitude of the resultant force is $\sqrt{(-25\sqrt{3})^2+125^2}=\boxed{50\sqrt{7}}$.