Geometry Solutions

1. (a) The volume of a cube with side length s is s^3. The volume of a rectangular prism with width w, length ℓ, and height h is $w\ell h$. Therefore,

$$(2\text{ in})^3 = (3\text{ in}) \cdot (1\text{ in}) \cdot h \Rightarrow h = \boxed{\frac{8}{3}\text{ in}}.$$

(b) The volume of a cone with radius r and height h is $\frac{\pi r^2 h}{3}$. The diameter of Jason's cone is 6 inches, so its radius is 3 inches. Thus, the volume of his cone is $\frac{\pi \cdot (3\text{ in})^2 \cdot (2\text{ in})}{3} = 6\pi\text{ in}^3$.

The volume of a cylinder with radius r and height h is $\pi r^2 h$. Thus, the volume of Jason's cylinder is $\pi\left(\frac{3}{2}\text{ in}\right)^2 \cdot (10\text{ in}) = \frac{45\pi}{2}\text{ in}^3$. As a result, Jason will need $\frac{45\pi}{2} - 6\pi = \boxed{\frac{33\pi}{2}\text{ in}^3}$ more water.

(c) From the statement of the problem it follows that the volume of the sphere is equal to the volume of the pyramid. The volume of a sphere with radius r is $\frac{4\pi r^3}{3}$. The volume of the sphere is $\frac{4\pi(1\text{ ft})^3}{3} = \frac{4\pi}{3}\text{ ft}^3$.

The volume of the square base pyramid with side length s and height h is $\frac{s^2 h}{3}$. (In general, the volume of a pyramid with base area B and height h is $\frac{1}{3}Bh$.) Therefore, the volume of Bart's pyramid is $\frac{(2\text{ ft})^2 h}{3} = \frac{4h}{3}\text{ ft}^2$. Thus, we have that

$$\frac{4\pi}{3} = \frac{4h}{3} \Rightarrow h = \boxed{\pi\text{ ft}}.$$

2. a) By the Pythagorean Theorem,

$$\begin{aligned}12^2 + (2x-1)^2 &= (2x+1)^2\\ 144 + 4x^2 - 4x + 1 &= 4x^2 + 4x + 1\\ 144 &= 8x\\ 18 &= x.\end{aligned}$$

The missing sides are $2x - 1 = \boxed{35}$ and $2x + 1 = \boxed{37}$.

b) By the Pythagorean Theorem,

$$\begin{aligned}x^2 + (2x-1)^2 &= 17^2\\ x^2 + 4x^2 - 4x + 1 &= 289\\ 5x^2 - 4x - 288 &= 0\\ (5x+36)(x-8) &= 0\\ x &= -\frac{36}{5}, 8.\end{aligned}$$

Since length cannot be negative, the only possible value of x is 8. The side lengths are $x = \boxed{8}$ and $2x - 1 = \boxed{15}$.

3. Using the distance formula and squaring both sides after the first step,

$$\begin{aligned}\sqrt{(12-a)^2 + (a-5)^2} &= \sqrt{(8-3)^2 + (16-4)^2}\\ (12-a)^2 + (a-5)^2 &= 5^2 + 12^2\\ 144 - 24a + a^2 + a^2 - 10a + 25 &= 169\\ 2a^2 - 34a + 169 &= 169\\ 2a(a-17) &= 0\\ a &= 0, 17\end{aligned}$$

Therefore, the sum of all possible values of a is $\boxed{17}$.

4. If a point with coordinates (a,b) is 1 away from $(0,1)$, the distance formula yields

$$\sqrt{a^2+(b-1)^2}=1 \Rightarrow a^2+(b-1)^2=1.$$

If (a,b) also lies on the unit circle, then $a^2+b^2=1$. We thus have a system of equations that we can solve by subtraction method:

$$\begin{cases} a^2+b^2=1 \\ a^2+(b-1)^2=1 \end{cases} \Rightarrow b^2-(b-1)^2=0 \Rightarrow b^2-b^2+2b-1=0 \Rightarrow 2b=1 \Rightarrow b=\frac{1}{2}.$$

We now substitute $b=\frac{1}{2}$ into the first equation to solve for a:

$$a^2+\left(\frac{1}{2}\right)^2=1 \Rightarrow a^2=\frac{3}{4} \Rightarrow a=\pm\frac{\sqrt{3}}{2}.$$

The possible points are $\boxed{\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)}$.

5. (a) The surface area of a cube with side length s is $6s^2$. The surface area of a rectangular prism with width w, length ℓ, and height h is $2(w\ell+h\ell+wh)$. Therefore,

$$\begin{aligned} 6(3\text{ cm})^2 &= 2((1\text{ cm})\cdot\ell+(7\text{ cm})\cdot\ell+(1\text{ cm})\cdot(7\text{ cm})) \\ 54\text{ cm}^2 &= 2(7\text{ cm}^2+(8\text{ cm})\ell) \\ 20\text{ cm}^2 &= (8\text{ cm})\ell \\ \ell &= \boxed{\frac{5}{2}\text{ cm}}. \end{aligned}$$

(b) The surface area of a cylinder with radius r and height h is $2\pi rh+2\pi r^2$. Since Doug's cylinder has diameter 2 feet, it has radius 1 foot. Thus, the surface area of Doug's cylinder is $2\pi(1\text{ ft})(7\text{ ft})+2\pi(1\text{ ft})^2=16\pi\text{ ft}^2$.

The surface area of a sphere with radius r is $4\pi r^2$. Doug's sphere has surface area

$4\pi(5\text{ ft})^2 = 100\pi\text{ ft}^2$. Therefore, Doug will need $100\pi\text{ ft}^2 - 16\pi\text{ ft}^2 = 84\pi\text{ ft}^2$. Since Doug needs 1 oz of paint to cover each additional square foot, he will need a total of $\boxed{84\pi\text{ oz}}$ of additional paint.

(c) The volume of a cone with radius r and height h is $\frac{\pi r^2 h}{3}$. Since the diameter of the cone is 8 mm, its radius is 4 mm. Thus, its volume is $\frac{\pi(4\text{ mm})^2(5\text{ mm})}{3} = \frac{80\pi}{3}\text{ mm}^3$. The surface area of a cylinder with height h and radius r is $2\pi rh + 2\pi r^2$. The surface area of the cylinder in the problem is $2\pi(3\text{ mm})(h) + 2\pi(3\text{ mm})^2 = 6\pi h\text{ mm} + 18\pi\text{ mm}^2$. Therefore,

$$\frac{80\pi}{3} = 6\pi h + 18\pi \Rightarrow \frac{26\pi}{3} = 6\pi h \Rightarrow h = \boxed{\frac{13}{9}\text{ mm}}.$$

6. (a) The area of a circle with radius r is πr^2, so the area of the unit circle is $\pi(1)^2 = \pi$. The area of an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$. Thus,

$$\frac{s^2\sqrt{3}}{4} = \pi \Rightarrow s^2 = \frac{4\pi}{\sqrt{3}} = \boxed{\frac{4\pi\sqrt{3}}{3}\text{ square units}}.$$

(b) By the Pythagorean Theorem, the hypotenuse of this right isosceles triangle is $\sqrt{2^2+2^2} = 2\sqrt{2}$. Therefore, its perimeter is $2+2+2\sqrt{2} = 4+\sqrt{2}$. The perimeter of an equilateral triangle with side s is $3s$. Since the perimeters of the two triangles are equal,

$$3s = 4+2\sqrt{2} \Rightarrow s = \frac{4+2\sqrt{2}}{3}.$$

The area of an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$, so the area of the equilateral triangle with side length $s = \frac{4+2\sqrt{2}}{3}$ is

$$\frac{(\frac{4+2\sqrt{2}}{3})^2\sqrt{3}}{4} = \frac{\frac{16+16\sqrt{2}+8}{9}\sqrt{3}}{4} = \boxed{\frac{24\sqrt{3}+16\sqrt{6}}{36}\text{ square units}}.$$

(c) The circumference of a circle with radius r is $2\pi r$, so the circumference of a circle with radius 3 is $2\pi(3) = 6\pi$. Let the length of a leg of this right isosceles triangle be s. By the Pythagorean Theorem, the length of the hypotenuse is then $\sqrt{s^2+s^2} = s\sqrt{2}$. Setting the perimeter of the triangle equal to the circumference of the circle yields

$$\begin{aligned} 2s + s\sqrt{2} &= 6\pi \\ s(2+\sqrt{2}) &= 6\pi \\ s &= \frac{6\pi}{2+\sqrt{2}} \text{ units.} \end{aligned}$$

The area of a triangle with base b and height h is $\frac{1}{2}bh$. The isosceles right triangle has base s and height s, so its area is $\frac{1}{2}\cdot s \cdot s$, or

$$\frac{1}{2}\left(\frac{6\pi}{2+\sqrt{2}}\right)^2 = \frac{1}{2}\cdot\frac{36\pi^2}{6+4\sqrt{2}} = \frac{9\pi^2}{3+2\sqrt{2}} = \frac{9\pi^2}{3+2\sqrt{2}}\cdot\frac{3-\sqrt{2}}{3-\sqrt{2}} = \boxed{27\pi^2 - 18\pi^2\sqrt{2}}.$$

Note that in the last step we rationalize the denominator by multiplying the fraction by $\frac{3-2\sqrt{2}}{3-2\sqrt{2}}$ and the new denominator is 1 since

$$(3+2\sqrt{2})(3-2\sqrt{2}) = 3^2 - (2\sqrt{2})^2 = 9-8 = 1.$$

7. (a) The length AB becomes the height of this cylinder, and both sides BC and AD become the radii of this cylinder. The volume of a cylinder with radius r and height h is $\pi r^2 h$. Since all sides of a square are equal, we have $r = h$, so the volume of our cylinder is

$$\pi r^2 h = 8\pi \Rightarrow \pi h^3 = 8\pi \Rightarrow h = 2 \text{ units.}$$

Since the height of the cylinder is equal to AB, we must have $AB = \boxed{2 \text{ units}}$.

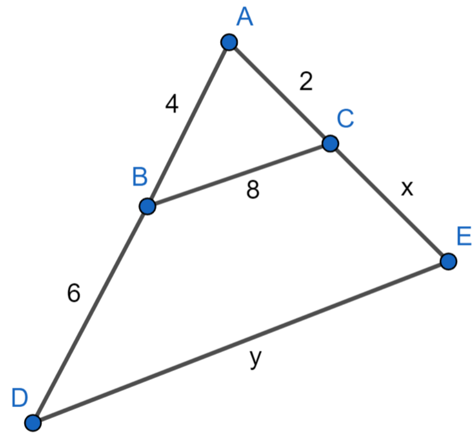
(b) The diameter of the smaller circle becomes the radius of the bigger circle. Therefore, the new circle has radius $2\cdot 2 = 4$, and area $\pi(4)^2 = \boxed{16\pi}$.

(c) The cone has height AB and radius BC. The volume of a cone with radius r and height h is $\frac{\pi r^2 h}{3} = \frac{\pi(AB)^2(BC)}{3} = \boxed{12\pi \text{ in}^3}$.

8. a) Since $DE \parallel BC$, $\angle ABC \cong \angle ADE$ and $\angle ACB \cong \angle AED$ by the Corresponding Angles Theorem, so $\triangle ABC \sim \triangle ADE$ by Angle-Angle similarity. Since corresponding sides of similar triangles are in proportion,

$$\frac{AE}{AC} = \frac{AD}{AB} \Rightarrow \frac{x+2}{2} = \frac{10}{4} \Rightarrow x+2=5 \Rightarrow \boxed{x=3}.$$

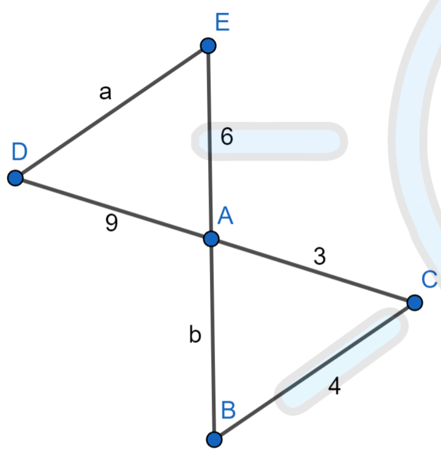
$$\frac{DE}{BC} = \frac{AE}{AC} \Rightarrow \frac{y}{8} = \frac{10}{4} \Rightarrow \boxed{y=20}.$$



b) Since $DE \parallel BC$, $\angle ADE \cong \angle ACB$ and $\angle AED \cong \angle ABC$ by the Alternate Interior Angles Theorem. Therefore, $\triangle ABC \sim \triangle AED$ by Angle-Angle Similarity. Since corresponding sides of similar triangles are in proportion,

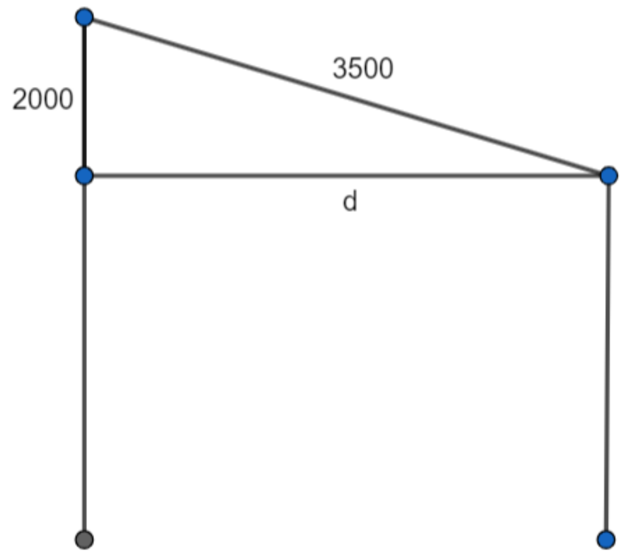
$$\frac{DE}{CB} = \frac{AD}{AC} \Rightarrow \frac{a}{4} = \frac{9}{3} \Rightarrow \boxed{a=12}.$$

$$\frac{AB}{AE} = \frac{AC}{AD} \Rightarrow \frac{b}{6} = \frac{3}{9} \Rightarrow \boxed{b=2}.$$



9. We can simplify the calculation by considering a smaller right triangle similar to the original one. A convenient similar triangle has legs 4 and $\frac{d}{500}$ and a hypotenuse of 7. Essentially, we have scaled down the triangle by a factor of 500. By the Pythagorean Theorem, we see that

$$\frac{d}{500} = \sqrt{7^2 - 4^2} = \sqrt{33} \Rightarrow \frac{d}{3500} = \boxed{\frac{\sqrt{33}}{7}}.$$



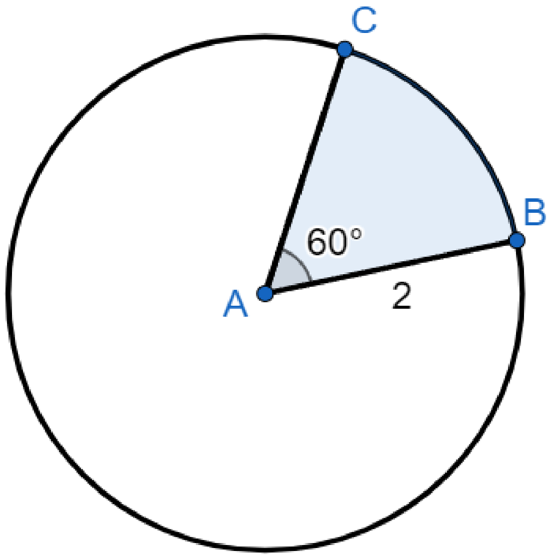
10. Let $AB = x$. Then $BC = 2x$ since $AB : BC = 1 : 2$. Since $AB : BE = 2 : 3$, we also have that $BE = \frac{3x}{2}$. The ratio of the areas of the rectangles is

$$\frac{AB \cdot BC}{AB \cdot BE} = \frac{BC}{BE} = \frac{2x}{\frac{3x}{2}} = \boxed{\frac{4}{3}}.$$

11. In a circle with radius r, the length of the arc subtended by a central angle equal to θ degrees is $\frac{\theta}{360^\circ} \cdot 2\pi r$ and the area of the sector subtended by θ is $\frac{\theta}{360^\circ} \cdot \pi r^2$. Applying these formulas in the diagram below yields:

$$\text{Arc Length} = \frac{60^\circ}{360^\circ} \cdot 2\pi(2\text{ cm}) = \boxed{\frac{2\pi}{3}\text{ cm}}.$$

$$\text{Sector Area} = \frac{60^\circ}{360^\circ} \cdot \pi(2\text{ cm})^2 = \boxed{\frac{2\pi}{3}\text{ cm}^2}.$$

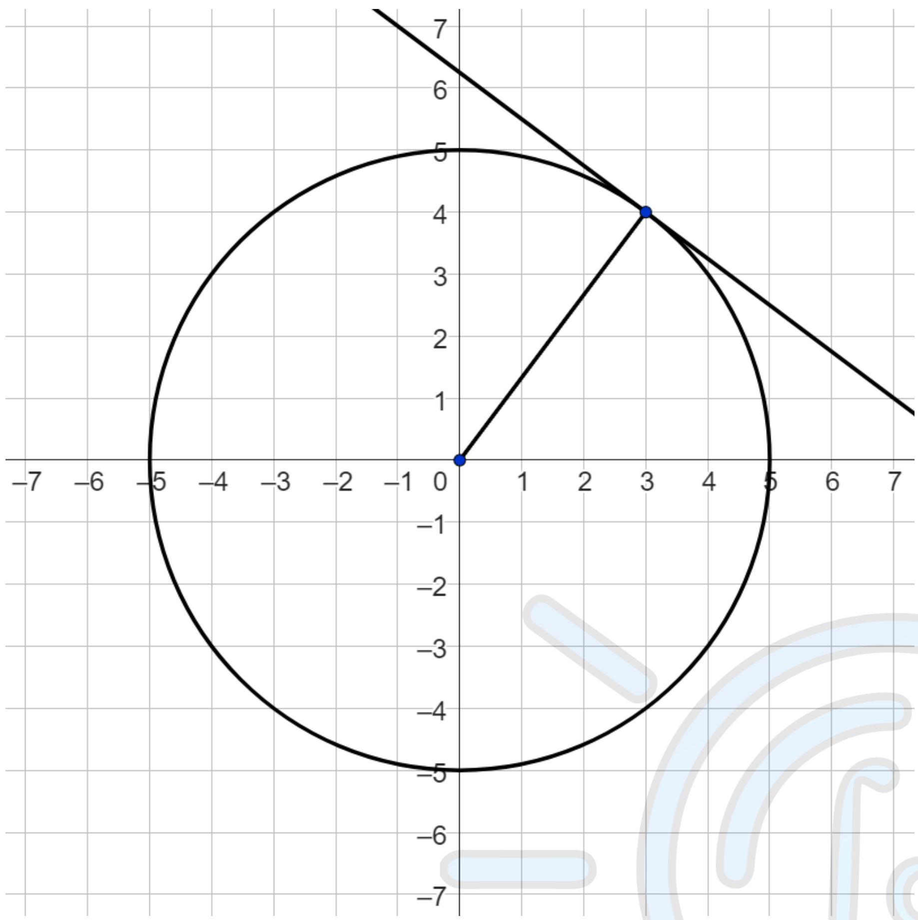


12. The segment connecting a center of the circle to the point of tangency is perpendicular to the tangent line. This segment contains the points $(0,0)$ and $(3,4)$, so the slope of the radius perpendicular to the tangent line is $\frac{4}{3}$. The slope of the tangent line is the negative reciprocal of $\frac{4}{3}$, which is $-\frac{3}{4}$. We now use point-slope equation of a line to find the equation of the tangent line through $(3,4)$:

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$4y - 16 = -3x + 9$$

$$\boxed{3x + 4y = 25}.$$



13. We wish to solve the system

$$\begin{cases} (x-2)^2 + (y-1)^2 = 4 \\ x^2 + (y-1)^2 = 9 \end{cases}$$

We may subtract the first equation from the second equation to cancel $(y-1)^2$. We then can solve the resulting equation for x:

$$\begin{aligned} &x^2 - (x-2)^2 = 5 \\ &x^2 - (x^2 - 4x + 4) = 5 \\ &4x - 4 = 5 \\ &x = \frac{9}{4}. \end{aligned}$$

Substituting $x = \frac{9}{4}$ into the second equation, we can solve for y:

$$\left(\frac{9}{4}\right)^2 + (y-1)^2 = 9 \Rightarrow (y-1)^2 = \frac{63}{16} \Rightarrow y - 1 = \pm\sqrt{\frac{63}{16}} \Rightarrow y = 1 \pm \frac{3\sqrt{7}}{4},$$

which yields the two points $\boxed{\left(\frac{9}{4}, 1 + \frac{3\sqrt{7}}{4}\right), \left(\frac{9}{4}, 1 - \frac{3\sqrt{7}}{4}\right)}$.

14. The surface area of the box with edge length s would be $5s^2$ because the box with an open top has only 5 faces. Therefore,

$$5s^2 = 150 \text{ in}^2 \Rightarrow s^2 = 30 \text{ in}^2 \Rightarrow s = \sqrt{30} \text{ in} \Rightarrow s^3 = \boxed{30\sqrt{30} \text{ in}^3}.$$

15. (a) Let the length of this rectangle be ℓ, and the width be w. We are given that $w\ell = 3 \text{ in}^2$. Therefore, if we double the length and triple the width, the area becomes

$$(2\ell)(3w) = 6(w\ell) = 6(3 \text{ in}^2) = \boxed{18 \text{ in}^2}.$$

(b) Let the radius of circle B be r_B. Let the radius of circle A be r_A. We are given that the area of circle B is 100 times the area of circle A, so $\pi r_B^2 = 100\pi r_A^2$. Dividing both sides by 100 and taking square roots of both sides yields

$$\pi r_B^2 = 100\pi r_A^2 \Rightarrow r_B^2 = 100 r_A^2 \Rightarrow r_B = 10 r_A \Rightarrow \frac{r_B}{r_A} = \boxed{10}.$$