Exponents and Logarithms Solutions

1. Using the properties of logarithms, we may write

$$\begin{aligned}\log\left(\frac{a^3b^4}{c^7}\right) &= \log\left(a^3b^4\right) - \log\left(c^7\right)\\ &= \log\left(a^3\right) + \log\left(b^4\right) - \log\left(c^7\right)\\ &= 3\log a + 4\log b - 7\log c\\ &= 3\cdot 5 + 4\cdot 2 - 7\cdot 1\\ &= \boxed{16}.\end{aligned}$$

2. (a) We use the difference-to-quotient formula $\log_c a - \log_c b = \log_c\left(\frac{a}{b}\right)$:

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = \boxed{4}.$$

(b) Because a^x and $\log_a x$ are inverse functions, we know that $a^{\log_a b} = b$. Thus

$$2^{\log_2 17} = \boxed{17}.$$

(c) We use the change of base formula $\log_b a = \dfrac{\log_c a}{\log_c b}$: Therefore,

$$\log_2 5\cdot\log_5 8 = \frac{\cancel{\log 5}}{\log 2}\cdot\frac{\log 8}{\cancel{\log 5}} = \frac{\log 8}{\log 2} = \log_2 8 = \boxed{3}.$$

3. (a) We isolate 7^x and then rewrite the expression in logarithmic form:

$$\begin{aligned}3\cdot 7^x + 1 &= 5\\ 3\cdot 7^x &= 4\\ 7^x &= \frac{4}{3}\\ x &= \boxed{\log_7\left(\frac{4}{3}\right)}.\end{aligned}$$

(b) Similarly to (a), we isolate e^x and rewrite the expression in logarithmic form:

$$\begin{aligned} 5e^x + 1 &= \frac{2e^x + 3}{2} \\ 10e^x + 2 &= 2e^x + 3 \\ 8e^x &= 1 \\ e^x &= \frac{1}{8} \\ x &= \boxed{\ln\left(\frac{1}{8}\right)}. \end{aligned}$$

(c) This equation is a quadratic in 2^x, so we may solve it by factoring:

$$\begin{aligned} &4^x - 3 \cdot 2^{x+1} + 5 = 0 \\ &(2^x)^2 - 6 \cdot 2^x + 5 = 0 \\ &(2^x - 5)(2^x - 1) = 0 \\ &2^x = 5 \text{ or } 2^x = 1 \\ &x = \boxed{\log_2 5} \text{ or } x = \boxed{0}. \end{aligned}$$

4. (a) The change of base formula $\log_b a = \dfrac{\log_c a}{\log_c b}$ yields

$$\frac{\log_2 y}{\log_2 5} = 3 \Rightarrow \log_5 y = 3 \Rightarrow y = 5^3 = \boxed{125}.$$

(b) The sum-to-product formula $\log_c a + \log_c b = \log_c (ab)$ yields

$$\log_2 y + \log_2 (y-4) = 2$$
$$\log_2 y(y-4) = \log_2 4$$
$$\log_2 (y^2 - 4y) = \log_2 4$$
$$y^2 - 4y = 4$$
$$y^2 - 4y - 4 = 0$$
$$y = \frac{4 \pm \sqrt{16+16}}{2} = 2 \pm 2\sqrt{2}.$$

Notice that we solved the quadratic using the Quadratic Formula. However, we are not done yet. Since logarithms are defined only for positive arguments, y and $y-4$ must be positive, meaning $y > 4$. Therefore, the only solution to the original equation is $y = \boxed{2 + 2\sqrt{2}}$.

(c) Here we have logarithms with different bases, so we must use the change of base formula $\log_b a = \dfrac{\log_c a}{\log_c b}$:

$$\log_4 19 = \frac{\log_2 19}{\log_2 4} = \frac{1}{2}\log_2 19 = \log_2 \sqrt{19}$$

Therefore,

$$\log_2(3y) = \log_2 \sqrt{19} \Rightarrow 3y = \sqrt{19} \Rightarrow y = \boxed{\frac{\sqrt{19}}{3}}.$$

5. (a) Rewriting both sides of the inequality in exponential form yields

$$\log_3 (4a - 7) \leq 3$$
$$4a - 7 \leq 3^3$$
$$4a \leq 34$$
$$a \leq \frac{17}{2}.$$

Because logarithms are defined only for positive arguments, we must have

$$4a - 7 > 0 \Rightarrow a > \frac{7}{4}.$$

Combining the two inequalities yields the interval $\boxed{\frac{7}{4} < a \leq \frac{17}{2}}$.

(b) Since both logarithms are to the same base, the inequality must hold for the arguments of the logarithms:

$$\log_7(5x-4) \geq \log_7(2x+8)$$
$$5x - 4 \geq 2x + 8$$
$$x \geq 4.$$

However, since logarithms are defined only for positive arguments, we must also have

$$5x - 4 > 0 \text{ and } 2x + 8 > 0 \Rightarrow x > \frac{4}{5} \text{ and } x > -4.$$

Combining the three inequalities $x \geq 4, x > \frac{4}{5}, x > -4$ results in a single inequality $\boxed{x \geq 4}$.

6. To find the y-intercept, set x equal to 0 and solve for y:

$$y = 5 \cdot 3^0 - 15 = \boxed{-10}.$$

To find the x-intercepts, set y equal to 0 and solve for x:

$$0 = 5 \cdot 3^{2x} - 15 \Rightarrow 5 \cdot 3^{2x} = 15 \Rightarrow 3^{2x} = 3 \Rightarrow 2x = 1 \Rightarrow x = \boxed{\frac{1}{2}}.$$

7. $f(x)$ is defined if $3x - 7 > 0 \Rightarrow x > \frac{7}{3}$ so the domain of $f(x)$ is $\boxed{\left(\frac{7}{3}, \infty\right)}$. Since the range of any logarithm function is all real numbers, the range of $f(x)$ is $\boxed{(-\infty, \infty)}$.

Another way to find the range of $f(x)$ is to find the domain of its inverse:

$$x = \log_3(3y-7)$$
$$3^x = 3y-7$$
$$y = \frac{3^x+7}{3}$$

It is clear that the domain of $f^{-1}(x)$ is all real numbers, hence, the range of $f(x)$ must be all real numbers.

8. (a) We observe that as x increases, the value of $\log_5(2x+1)$ also increases. Therefore, $h(x)$ is $\boxed{\text{increasing}}$.

   (b) We observe that as x increases, the value of $\log_7(-2x+15)$ decreases (because the argument of the logarithm decreases). Therefore, $g(x)$ is $\boxed{\text{decreasing}}$.

   (c) Notice that $3^{-x} = \frac{1}{3^x}$, meaning that as x increases, the denominator of the fraction increases and the value of the fraction decreases. Therefore, as x increases, $3^{-x}-2$ decreases, so $f(x)$ is $\boxed{\text{decreasing}}$.

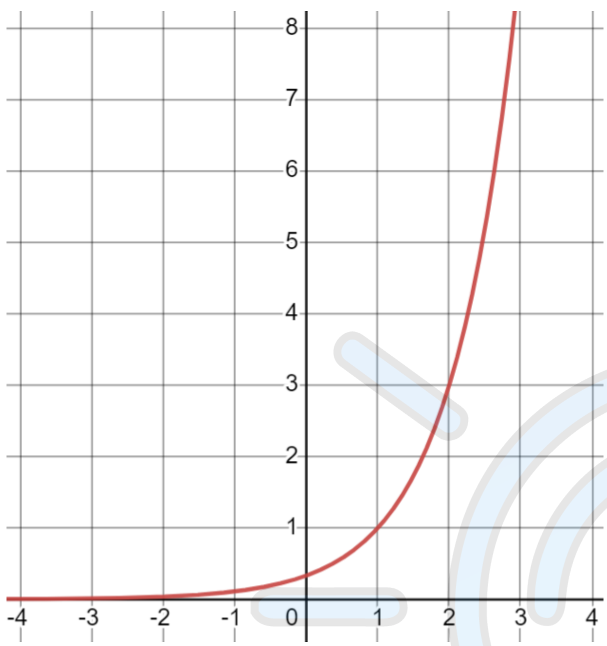
9. To find the inverse of $k(x)$, we replace $k(x)$ with x and x with y and solve for y in terms of x:

$$x = \frac{1}{e^y+1}$$
$$e^y+1 = \frac{1}{x}$$
$$e^y = \frac{1}{x}-1$$
$$y = \ln\left(\frac{1}{x}-1\right)$$
$$\boxed{k^{-1}(x) = \ln\left(\frac{1}{x}-1\right)}.$$

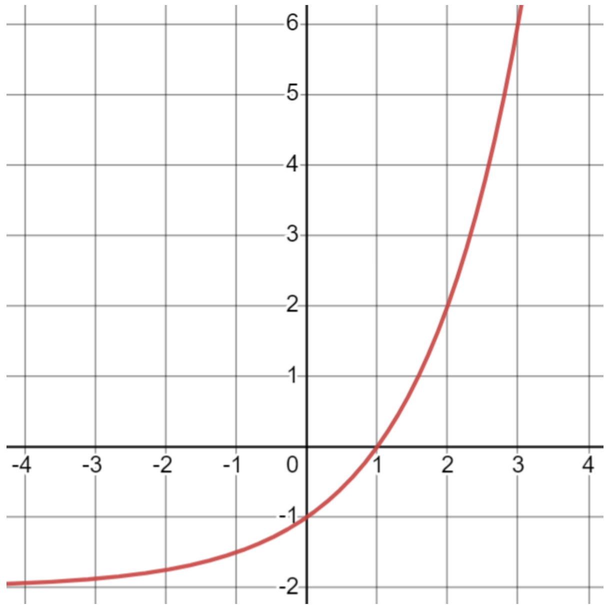
10. To find the inverse of $p(x)$, we replace $p(x)$ with x and x with y and solve for y in terms of x. We make use of the logarithm property $\log_b a^n = n \log_b a$. In the third step, we rewrite the equation in exponential form.

$$x = \log_7 (2y+1)^5$$
$$x = 5\log_7(2y+1)$$
$$\frac{x}{5} = \log_7(2y+1)$$
$$2y+1 = 7^{\frac{x}{5}}$$
$$y = \frac{7^{\frac{x}{5}}-1}{2}$$
$$\boxed{p^{-1}(x) = \frac{7^{\frac{x}{5}}-1}{2}}.$$

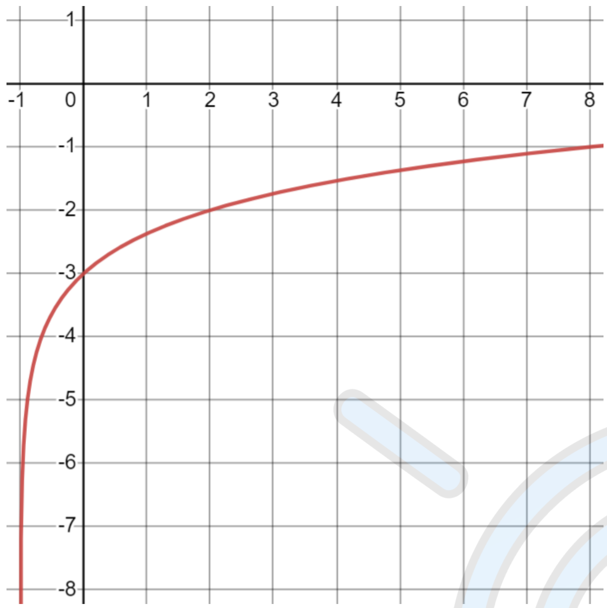
11. (a) We graph $y = 3^x$ by using a table of values to plot several points. To find the graph of $y = 3^{x-1}$, we shift the graph of $y = 3^x$ by 1 unit to the right:



(b) We graph $y = 2^x$ by using a table of values to plot several points. Then, we obtain the graph of $y = 2^x - 2$ by translating the graph of $y = 2^x$ by 2 units down:



(c) We graph $y = \log_2 x$ by using a table of values to plot several points. Then, we obtain the graph of $y = \log_2(x+1) - 3$ by translating the graph of $y = \log_2 x$ by 1 unit to the left and 3 units down:



We use the identities $\log_c ab = \log_c a + \log_c b$, $\log_c \frac{a}{b} = \log_c a - \log_c b$, and $\log_c a^b = b \log_c a$ along with the laws of exponents. Recall that $\sqrt[m]{a^n} = a^{n/m}$.

12. (a) $\ln\left(a^2 b^3\right) = \ln a^2 + \ln b^3 = \boxed{2\ln a + 3\ln b}$.

(b) $\log(abc)^{\frac{5}{2}} = \frac{5}{2}\log abc = \frac{5}{2}(\log a + \log b + \log c) = \boxed{\frac{5}{2}\log a + \frac{5}{2}\log b + \frac{5}{2}\log c}$.

(c) Notice that $\log 100 = \log_{10} 100 = 2$.

$$\begin{aligned}\log \frac{100\sqrt[3]{a}}{b^4\sqrt{c^5}} &= \log 100 + \log \sqrt[3]{a} - \log b^4 - \log \sqrt{c^5} \\ &= \log 100 + \log a^{1/3} - \log b^4 - \log c^{5/2} \\ &= \boxed{2 + \frac{1}{3}\log a - 4\log b - \frac{5}{2}\log c}.\end{aligned}$$

13. Because the problem specifies half-life, we know that the model for the problem will be exponential decay. Hence, the amount of caffeine in Manuel's body after t hours can be modeled by the function $P(t) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5}}$. The exponent of $\frac{1}{2}$ is $\frac{t}{5}$ because every 5 hours the total amount of caffeine in the bloodstream is halved, or multiplied by $\frac{1}{2}$. We wish to find the time t when $P(t) = 20$, so we solve

$$\begin{aligned}P(t) &= 20 \\ 200 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5}} &= 20 \\ \left(\frac{1}{2}\right)^{\frac{t}{5}} &= \frac{1}{10} \\ \frac{t}{5} &= \log_{\frac{1}{2}} \frac{1}{10} \\ t &= 5\log_{\frac{1}{2}} \frac{1}{10} \approx \boxed{16.610 \text{ hours}}.\end{aligned}$$

14. The amount of money that James has after t years can be modeled by the exponential equation $P(t) = 800,000 \cdot (1.1)^t$. This ensures that after every year, the amount of money he has is multiplied by 1.1. In other words, the annual growth factor is 1.1. Setting $P(t)$ equal to 1,000,000 and rewriting exponential equation in logarithmic form in step 3, we get:

$$\begin{aligned}800,000 \cdot (1.1)^t &= 1,000,000 \\ (1.1)^t &= \frac{5}{4} \\ t &= \log_{1.1} \frac{5}{4} \approx \boxed{2.341 \text{ years}}.\end{aligned}$$

15. The population of rabbits after t years can be modeled by the exponential equation $P(t) = 1500 \cdot 2^t$. Note that the annual growth factor is 2. We wish to find the time t when $P(t) = 1,000,000$, so we solve

$$\begin{aligned} P(t) &= 1,000,000 \\ 1500 \cdot 2^t &= 1,000,000 \\ 2^t &= \frac{2000}{3} \\ t &= \log_2 \frac{2000}{3} \approx \boxed{9.381 \text{ years}}. \end{aligned}$$